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Laminar boundary layer in low Prandtl number flows with variable thermal properties on a flat plate

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SUMMARY

The laminar boundary layer flow of a low Prandtl number fluid with arbitrary thermal properties past a flat plate is studied by the method of matched asymptotic expansions. By assuming power law relations for the viscosity, density and Prandtl number, the first order results for the skin friction, the recovery factor and the heat transfer rate at the wall are obtained. It turns out that the outer flow in the thermal boundary layer is governed by a simple nonlinear differential equation of second order, which is correct to all orders in Prandtl number. Exact and approximate solutions to this outer equation are obtained. Further, it is shown that the first order terms for the recovery factor are independent of the thermal properties, while the heat transfer terms have a complicated dependence. The skin friction result shows the dependence on thermal properties, Mach number and heat transfer rate.

1. Introduction

The problem of the compressible boundary layer on a flat plate at a given Mach number depends upon two parameters of the fluid: the Prandtl number and the viscosity-temperature relationship. The problem has been studied extensively in the past and good reviews of the results available at present have been made by Stewartson [1] and Lagerstrom [2]. Except for certain specific numerical calculations, all the available works invariably assume that the viscosity μ is proportional to the temperature T and that the Prandtl number σ is constant ([1, 2]). For most of the gases, however, the viscosity-temperature relationship is better represented either by a simple power law $\mu \propto T^w$, where w is a constant usually between $\frac{1}{2}$ and 1, or by the Sutherland law which involves two constants and so is somewhat more complicated.

In the present work we aim to study the case of a flow of general variable thermal properties at low Prandtl numbers. The problem of low Prandtl number is of interest for liquid metals. For air the Prandtl number is 0.73 at normal temperature and pressures, but flights at high speeds and altitudes lead to high temperatures and low pressures at which the Prandtl number of the air can become very small [3]. Moreover, it is well-known [4] that the studies of such asymptotic solutions are exceedingly useful from a practical point of view, owing to the fact that in many situations these have been found to hold with surprising accuracy even under distinctly non-asymptotic conditions. Edward and Tellep [5] have analysed the heat transfer with variable thermal properties by ignoring the viscous dissipation. To account for the variation of thermal properties with temperature, these authors have used for the term $\rho\mu/\sigma$ in the energy equation a power law dependence of the type T^m. They have solved the problem for a range of m between zero and unity ($0 \leqq m \leqq 1$). However, for a perfect gas and for most liquids, m is usually between zero and minus unity ($-1 \leqq m \leqq 0$). It is for this range that we study here the heat transfer problem taking full account of dissipation. Furthermore, the case of the insulated wall is also studied for arbitrary fluid properties.

To study the problem of low Prandtl numbers systematically we have used the method of matched asymptotic expansions. In this method we usually study the inner and outer limits of the equations and try to match them in an overlap domain. It turns out that the outer flow is governed by a simple nonlinear equation of second order, which is correct to all orders in Prandtl number. When the wall is thermally insulated, the simple solution T= constant of this outer flow equation (correct to zeroth order in Prandtl number) matches with the corresponding inner solution which simplifies the problem considerably. For the heat transfer problem numerical solutions for this outer equation are carried out yielding an outer solution correct to all orders in Prandtl number. However, an approximate solution is also given in the Appendix and compared with the exact results. Edward and Tellep [5] have solved what we call here the zeroth order outer equation; this solution is, however, uniformly valid up to the order $\sigma^{\frac{1}{2}}$ for the heat transfer case. To get higher order terms it is essential to formulate a different inner limit for the equations.

2. Governing equations

The boundary layer equations for the steady flow of a compressible fluid past a flat plate can be written in the usual notations [1] as

$$(Nf'')' + ff'' = 0\,, \tag{1}$$

$$\left(\frac{N}{\sigma}T'\right)' + fT' + CNf''^2 = 0\,. \tag{2}$$

Here f' is the longitudinal velocity u at any point as a fraction of the free stream value U_∞; T is the temperature as a fraction of the free stream value T_∞.

The Prandtl number σ and the quantities N and C are defined by

$$\sigma = \mu C_p/K\,, \quad N = \rho\mu/\rho_\infty\mu_\infty\,, \quad C = (\gamma-1)M_\infty^2\,,$$

where C_p is the specific heat, K the thermal conductivity, γ the ratio of specific heat and M_∞ the free stream Mach number. Dashes on f and T denote the differentiation with respect to Howarth–Dorodnitsyn variable

$$\eta = \left(\frac{U_\infty\rho_\infty}{2\mu_\infty x}\right)^{\frac{1}{2}} \int_0^y (\rho/\rho_\infty)dy\,,$$

where x is the distance along the plate measured from the leading edge and y is the distance normal to it.

The boundary conditions on the velocity and temperature profiles are

$$f(0) = 0 = f'(0), \quad f'(\infty) = 1\,, \tag{3a}$$

$$T(0) = T_w \text{ or } T'(0) = 0;\ T(\infty) = 1\,. \tag{3b}$$

In (3b), only one of the two conditions at the wall is required; the first prescribes the wall temperature at a constant value; the second implies an insulated plate.

To account for the variation of fluid properties with temperature,we use the power law relations

$$\mu/\mu_\infty = T^w\,, \quad \rho/\rho_\infty = T^a, \quad \sigma/\sigma_\infty = T^b\,. \tag{4}$$

For a given fluid, the exponents w, a and b are to be determined from curve fitting to experimental data or by some theoretical consideration. Eliminating ρ, μ and σ in favour of T from (1) and (2) we get

$$f''' + f''[fT^{-(a+w)} + (a+w)T'/T] = 0\,,$$

$$T'' + T'[\sigma_\infty f T^{b-a-w} + (a+w-b)T'/T] + C\sigma_\infty f''^2 T^b = 0\,.$$

Further, introducing the transformation⋆

$$T = h^{m+1}\,, \quad m = \frac{b-a-w}{1+a+w-b} \tag{5}$$

the above equations reduce to

⋆ The symbol m defined in (5) does not denote the same quantity as in Edward and Tellep [5].

$$f''' + f''(fh^n - nh'/h) = 0\,. \tag{6}$$

$$h'' + \sigma_\infty f\, h^m h' + \sigma_\infty E h^{-n} f''^2 = 0\,, \tag{7}$$

where

$$n = -(a+w)(1+m), \quad E = C/(1+m)\,. \tag{8}$$

The boundary conditions for the velocity profile are the same as (3a) and for the temperature profile they are

$$h(0) = h_w = T_w^{1/(1+m)} \text{ or } h'(0) = 0,\ h(\infty) = 1\,. \tag{9}$$

3. Analysis at low Prandtl number

We now solve the equations (6) and (7) under the boundary conditions (3a) and (9) at small values of σ_∞ by the method of matched asymptotic expansions.

3.1. *Inner solution*

The inner limit is defined as $\sigma_\infty \to \sigma$ with η fixed, and we write

$$f = \sum_{s=0} f_s \sigma_\infty^{s/2}, \quad h = \sum_{s=0} h_s \sigma_\infty^{s/2}\,. \tag{10}$$

Substituting the above series (10) in equations (6) and (7), and collecting the coefficients of various powers of $\sigma_\infty^{\frac{1}{2}}$, the momentum equation (6) gives

$$f_0''' + f_0''(f_0 h_0^n - n h_0'/h_0) = 0\,, \tag{11a}$$

$$f_1''' + f_1''(f_0 h_0^n - n h_0'/h_0) + f_0''(f_1 h_0^n + n f_0 h_0^{n-1} h_1 - n h_1'/h_0 + n h_0' h_1/h_0^2) = 0\,, \tag{11b}$$

and the energy equation (7) gives

$$h_0'' = 0 \tag{12a}$$

$$h_1'' = 0 \tag{12b}$$

$$h_2'' = -f_0 h_0^m h_0' - E f_0''^2/h_0^n \tag{12c}$$

$$h_3'' = -h_0^m[f_1 h_0' + f_0 h_1' + m f_0 h_0' h_1/h\ ] + E[-2f_0'' f_1'' + n f_0''^2 h_1/h_0]/h_0^n\,. \tag{12d}$$

The solutions of the first three equations (12a, b, c) are

$$h_0 = a_0\eta + A_0\,, \tag{13a}$$

$$h_1 = a_1\eta + A_1\,, \tag{13b}$$

$$h_2 = -a_0 \int_0^\eta (\eta-\eta_1)(a_0\eta_1 + A_0)^m f_0(\eta_1)\,d\eta_1$$

$$-E \int_0^\eta (\eta-\eta_1)(a_0\eta_1 + A_0)^{-n} f_0''^2(\eta_1)\,d\eta_1 + a_2\eta + A_2\,. \tag{13c}$$

Here the a's and A's are constants of integration, either of them to be determined from inner boundary conditions.

The solutions (13) are singular for large η and do not satisfy the boundary conditions at infinity. This singularity is rather similar to the one-encountered in improving Stokes' solution in low Reynolds number flow.

3.2. *Outer solution*:

We now need a different outer limit. From an order of magnitude analysis we introduce the outer variables

$$t = \sigma_\infty^{\frac{1}{2}} \eta, \quad F(t) = \sigma_\infty^{\frac{1}{2}} f(\eta), \quad H(t) = h(\eta) \tag{14}$$

and study the limit $\sigma_\infty \to 0$ with t, F and H fixed. With this outer limit the equations (6) and (7) become

$$F F_{tt} H^n + \sigma_\infty (F_{ttt} - n F_{tt} H_t / H) = 0 \,, \tag{15}$$

$$H_{tt} + F H^m H_t + \sigma_\infty E F_{tt}^2 H^{-n} = 0 \,. \tag{16}$$

It can be shown very easily that the solution correct to all orders in σ_∞ of the momentum equation (15) which satisfies the boundary conditions at infinity is

$$F = t - \sigma_\infty^{\frac{1}{2}} \beta + O(\sigma_\infty^\infty) \,, \tag{17}$$

where β is a constant independent of t but a function of σ_∞, say

$$\beta = \sum_{s=0} \beta_s \sigma_\infty^{s/2} \tag{17a}$$

and σ_∞^∞ is used to denote exponentially small terms in the limit $\sigma_\infty \to 0$. Now with the aid of (17) the outer energy equation (16) may be written as

$$H_{tt} + (t - \sigma_\infty^{\frac{1}{2}} \beta) H^m H_t = O(\sigma_\infty^\infty) \,. \tag{18}$$

The outer boundary condition is $H(\infty) = 1$. This nonlinear outer equation (18) is correct to all orders in σ_∞, i.e., the error is exponentially small. This is due to the fact that at low Prandtl numbers, there is a thin momentum boundary layer inside a thick thermal boundary layer and which has the effect of displacing the stream lines from their inviscid position by an amount β in the η-scale. The outer equations of all orders therefore follow from this simple equation (18) which can be solved once and for all. For $m = 0$ the solution (correct to all orders in σ_∞) to the equation (18) which satsfies the boundary condition at infinity is

$$H = D + (1 - D) \operatorname{erf}[(t - \sigma_\infty^{\frac{1}{2}} \beta)/\sqrt{2}] \tag{19}$$

where $\operatorname{erf}(x)$ is the well-known error function defined by

$$\operatorname{erf}(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x e^{-p^2} dp$$

and D is a constant independent of t but a function of σ_∞, say

$$D = \sum_{s=0} D_s \sigma_\infty^{s/2} \,.$$

It appears that for a nonzero value of m, the equation (18) has, in general, to be solved numerically. However, for this case ($m \neq 0$) it is interesting to note that when wall is thermally insulated a simple solution of the outer equation (18) is $H(t) = 1$ (correct to zeroth order in σ_∞) which matches with the corresponding inner solution. This means that in the case of an insulated wall a few lower order terms can be obtained without solving equation (18) numerically. Therefore, we study the equation (18) separately for the cases of the insulated wall and the heat transfer.

3.2.1. *Insulated wall*

Before we solve nonlinear outer equation (18) let us note that for an insulated wall $h'(0) = 0$ and the solutions of the inner equations (14) give $a_0 = a_1 = a_2 = 0$. Now the outer expansion of the inner solution (13) for large η may be written as

$$h = A_0 + \sigma_\infty^{\frac{1}{2}} \left[A_1 + t E A_0^{-n} \int_0^\infty f_0''^2 d\eta \right] + O(\sigma_\infty) \,. \tag{20}$$

We now proceed to solve the outer equation (18). Let us assume

$$H = \sum_{s=0} H_s \sigma_\infty^{s/2} . \tag{21}$$

Substituting this expansion (21) and (17a) in equation (18) and collecting the coefficients of equal powers of $\sigma_\infty^{\frac{1}{2}}$, we get

$$H_{0tt} + tH_0^m H_{0t} = 0 , \tag{22a}$$

$$H_{1tt} + tH_0^m(H_{1t} + mH_{0t}H_1/H_0) = \beta_0 H_0^m H_{0t} , \tag{22b}$$

$$\begin{aligned} H_{2tt} + tH_0^m(H_{2t} + mH_{0t}H_2/H_0) = \\ = H_0^m[\beta_0 H_{1t} + \beta_0 m H_{0t} H_1/H_0 + \beta_1 H_{0t} - tmH_1H_{1t}/H_0 \\ - m(m-1)tH_{0t}H_1^2/(2H_0^2)] . \end{aligned} \tag{22c}$$

The corresponding boundary conditions at infinity are

$$H_0(\infty) = 1, \quad H_1(\infty) = 0, \quad H_2(\infty) = 0, \dots . \tag{22d}$$

The solution of (22a) which satisfies the boundary condition $H_0(\infty) = 1$ and matches with (20) up to zeroth order is

$$H_0(t) = 1 = A_0 . \tag{23}$$

The solution to the first order equation (22b) with boundary condition $H_1(\infty) = 0$ is

$$H_1 = D_1[1 - \text{erf}(t/\sqrt{2})] , \tag{24}$$

The inner expansion of (24) for small t is

$$H_1 = D_1 - t(2/\pi)^{\frac{1}{2}} D_1 + \dots . \tag{25}$$

Matching (25) with first order terms in (20), we get

$$A_1 = D_1 = E(\pi/2)^{\frac{1}{2}} \int_0^\infty f_0''^2 d\eta . \tag{26}$$

Now with the help of (23) and the solution of the outer momentum equation (17), the zeroth order inner momentum equation (11a) reduces to the Blasius equation with the solution, say $f_0(\eta) = \beta(\eta)$, and the first order equation (11b) has the solution

$$f_1 = nA_1(\eta B' - B)/2 .$$

Now the solution (17) of the outer momentum equation becomes

$$F = t - 1.21678\sigma_\infty^{\frac{1}{2}} - 0.28153(a+w)(\gamma-1)M_\infty^2 \sigma_\infty + O(\sigma_\infty^{\frac{3}{2}}) .$$

The wall recovery temperature is

$$T_r = 1 + C(\pi\sigma_\infty/2)^{\frac{1}{2}} \int_0^\infty B''^2 d\eta + O(\sigma_\infty) \tag{27}$$

and the recovery factor is

$$\begin{aligned} r &= (2\pi\sigma_\infty)^{\frac{1}{2}} \int_0^\infty B''^2 d\eta + O(\sigma_\infty) \\ &= 0.9255\sigma_\infty^{\frac{1}{2}} + O(\sigma_\infty) . \end{aligned} \tag{28}$$

The coefficient of skin friction C_f is given by

$$\begin{aligned} C_f\sqrt{R_x} &= [2f''\rho\mu/\rho_\infty\mu_\infty]_{\eta=0} \\ &= 0.6641 + 0.1530(a+w)(\gamma-1)M_\infty^2\sigma_\infty^{\frac{1}{2}} + O(\sigma_\infty) , \end{aligned} \tag{29}$$

where R_x is the local Reynolds number based upon distance x.

3.2.2. *Heat transfer*

The nonlinear outer equation (18) can actually be solved once and for all. If

$$z = t - \beta\sigma_{\infty}^{\frac{1}{2}}, \tag{30}$$

then the outer equation becomes

$$H_{zz} + zH^m H_z = 0. \tag{31}$$

This outer equation (31) has been solved numerically (on the IBM 7044 Computer at IIT Kanpur) to yield the solution which is also correct to all orders in σ_{∞}. The method of solution is the same as that of Edward and Tellep [5]. Solutions are obtained for a range of m between zero and unity ($0 \leq m \leq 1$). The results of the numerical solution necessary for matching are shown in Fig. 1, which displays $H_z(0)$ *vs.* $H(0)$ for various values of m.

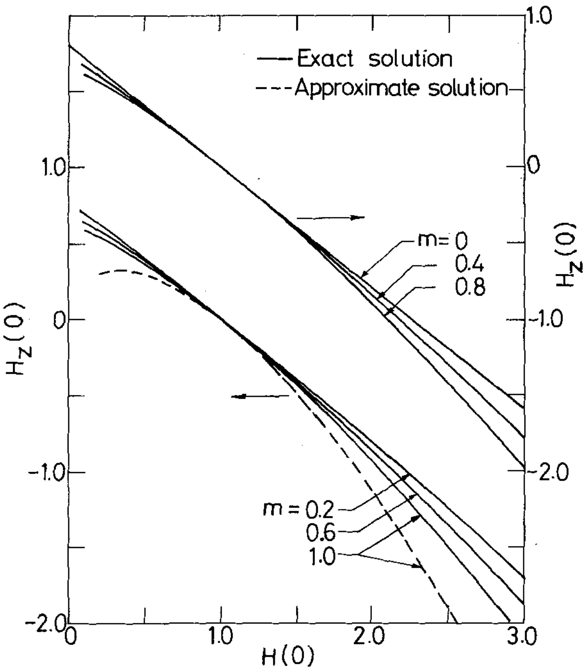

Figure 1. Solution of outer equation (31).

To affect the matching we formally write the inner expansion of the (numerical) outer solution at $t \to 0$.

$$H = H(0) + \sigma_{\infty}^{\frac{1}{2}}(\eta - \beta)H_z(0) + O(\sigma_{\infty}^{\frac{3}{2}}). \tag{32}$$

Here $H(0)$ and $H_z(0)$ are constants independent of z but functions of σ_{∞}, say

$$H(0) = \sum_{s=0} D_s \sigma_{\infty}^{s/2},\ H_z(0) = \sum_{s=0} C_s \sigma_{\infty}^{s/2}. \tag{33}$$

Substituting (33) and (17a) in (32) we get the inner expansion of the outer solution

$$H = D_0 + \sigma_{\infty}^{\frac{1}{2}}(D_1 - \beta_0 C_0 + \eta C_0) + \sigma_{\infty}(D_2 - \beta_1 C_0 - \beta_0 C_1 + \eta C_1) + O(\sigma_{\infty}^{\frac{3}{2}}) \text{ as } \eta \to \infty \tag{34}$$

For the heat transfer case we prescribe the wall temperature $h(0) = h_w$, which requires from the inner solution (13), $A_0 = h_w$, $A_1 = A_2 = 0$. First, matching the zeroth order inner solution (13a) with zeroth order terms in (34), we get $a_0 = 0$. Now the outer expansion of the inner solution (13) for large η may be written as

$$h=h_w+\sigma_\infty^{\frac{1}{2}}a_1\eta+\sigma_\infty[a_2\eta+Eh_w^{-n}\int_0^\eta(\eta-\eta_1)f_0''^2(\eta_1)d\eta_1]+O(\sigma_\infty^{\frac{3}{2}}). \tag{35}$$

Matching (34) and (35) we get

$$a_0=0,\quad D_0=h_w$$
$$a_1=C_0,\quad D_1=\beta_0C_0$$
$$a_2=C_1+Eh_w^{-n}\int_0^\infty f_0''^2d\eta$$
$$D_2=\beta_0C_1+\beta_1C_0-Eh_w^{-n}\int_0^\infty \eta f_0''^2d\eta. \tag{36}$$

Using the matching results (36) and the solution of the outer equation (17), the solution of the zeroth inner momentum equation (11a) is

$$f_0(\eta)=h_w^{-n/2}B(\eta h_w^{n/2}),$$

while the first order equation (11b), is solved numerically by the Runge–Kutta method to give

$$f_0''(0)=-0.2857na_1/h_w,\quad f_1(\infty)=-0.2468na_1/h_w^{1+n}.$$

The solution (17) of the outer momentum equation now becomes

$$F=t-1.21678(T_w^{(a+w)/2}-T_r^{(a+w)/2})\sigma_\infty^{\frac{1}{2}}+0.2468(a+w)\alpha T_w^{a+w-1}\sigma_\infty^{\frac{1}{2}}$$
$$-1.21678\sigma_\infty^{\frac{1}{2}}-0.28153(a+w)(\gamma-1)M_\infty^2\sigma_\infty+O(\sigma_\infty^{\frac{3}{2}}). \tag{37}$$

The heat transfer rate at the wall is

$$h'(0)=C_0\sigma_\infty^{\frac{1}{2}}+[C_1+Eh_w^{-n/2}\int_0^\infty B''^2(\eta_1)d\eta_1]\sigma_\infty+\dots. \tag{38}$$

Introducing (5) and the recovery temperature (27), we get

$$T'(0)=[(m+1)T_w^{m/(1+m)}H_z(0)+(2/\pi)^{\frac{1}{2}}(T_r-1)T_w^{(2b-a-w)/2}]\sigma_\infty^{\frac{1}{2}}+O(\sigma_\infty). \tag{39}$$

For $m=0$, making use of (19), the heat transfer rate (39) becomes

$$T'(0)=(2\sigma_\infty/\pi)^{\frac{1}{2}}[1-T_w+(T_r-1)T_w^{b/2}]+O(\sigma_\infty).$$

If the Prandtl number is constant ($b=0$), we get the result which is due to Stewartson [1].

To study the heat transfer (39), in detail we rewrite it as

$$T'(0)=A+\alpha+O(\sigma_\infty), \tag{40}$$

where

$$A=(2\sigma_\infty/\pi)^{\frac{1}{2}}(T_r-1)T_w^{(2b-a-w)/2}$$
$$=0.3692(\gamma-1)\sigma_\infty M_\infty^2T_w^{(2b-a-w)/2} \tag{41}$$

is the contribution to heat transfer due to Mach number and

$$\alpha=\sigma_\infty^{\frac{1}{2}}(m+1)H_z(0)T_w^{m/(1+m)} \tag{42}$$

is due to the prescribed wall temperature. The quatntity $\alpha/\sigma_\infty^{\frac{1}{2}}$ is displayed in Fig. 2 for various values of T_w and m and is determined by the following procedure. For a given T_w and m, we first determine $H(0)=T_w^{1/(1+m)}$. From Fig. 1, we obtain $H_z(0)$ for a given $H(0)$ and m.

The skin friction coefficient is

$$C_f\sqrt{R_x}=0.6641(T_w^{(a+w)/2}-T_r^{(a+w)/2})-0.4040\alpha(a+w)/T_w$$
$$+0.6641+0.1530(a+w)(\gamma-1)M_\infty^2\sigma_\infty^{\frac{1}{2}}+O(\sigma_\infty). \tag{43}$$

When the wall is insulated, the above results (43) reduces to (29)

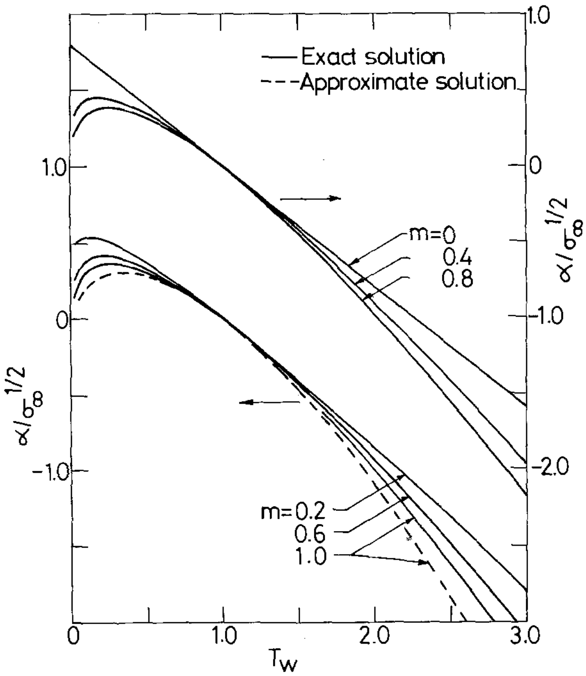

Figure 2. Heat transfer contribution α due to prescribed wall temperature.

4. Discussion

The recovery factor result (28) up to the order $\sigma_\infty^{\frac{1}{2}}$ is independent of the thermal properties.

Further, the approximate solution of the outer equation (A-2) leads to the recovery factor result which is identical to (28). It is interesting to note that this result (28) is the same as that obtained by various authors previously, when either $\rho\mu$ is constant or when the flow properties are independent of the temperature. It seems that the precise dependence of ρ, μ and σ on the temperature is not crucial when the Prandtl number is low. This is plausible, because the plate is thermally insulated and the fluid is highly conducting, therefore, the variation of the temperature across the boundary layer is small. Incidently, when the Prandtl number is of order unity the precise description of thermal properties is also not crucial since Kuerti's review [6] shows that for an insulated wall the temperature variation throughout the boundary layer is approximately the same function of velocity for various viscosity laws.

The skin friction result (29) for an insulated wall shows that the leading term is the Blasius value, while the first order term shows dependence on thermal properties and free stream Mach number. On the other hand, for the heat transfer case the skin friction result (43), even to zeroth order, depends upon thermal properties and wall temperature, while the first order terms depend also upon the heat transfer.

The heat transfer result (39) is a complicated function of the fluid properties, Mach number and wall temperature. The contribution A (41) to the heat transfer due to compressibility is proportional to $M_\infty^2\sigma_\infty$, while the contribution α (42) due to prescribed wall temperature is shown in Fig. 2. It is seen from the Fig. 2 that an increase of m decreases $|\alpha/\sigma_\infty^{\frac{1}{2}}|$ for $T_w < 1$, increases it for $T_w > 1$ and remains zero if $T_w = 1$. The approximate solution to the outer equation (A-2) yields an exact solution when $m = 0$ and for other values of m the error increases as m increases. To study the error in the approximate solution we consider the case of largest error (greatest m in our calculations i.e. $m = 1$) and plot $\alpha/\sigma_\infty^{\frac{1}{2}}$ (A-4) as shown in Fig. 2. It is seen that the approximate solution underestimates $|\alpha/\sigma_\infty^{\frac{1}{2}}|$ for $T_w < 1$, overestimates for $T_w > 1$ and is exact for $T_w = 1$. This under- and overestimation increases we go away from $T_w = 1$. Furthermore, it is clear from Fig. 2 that the result of the approximate analysis is good for $0.7 < T_w < 1.3$, the range in which the low Prandtl number analysis is most useful.

Appendix

Approximation solution

The outer equation (31) may formally be converted into an integral equation which satisfies the boundary condition at infinity

$$H = D + (1-D) \int_0^z e^{-\phi} dz \Big/ \int_0^\infty e^{-\phi} dz \qquad \text{(A-1)}$$

where

$$\phi(z) = \int_0^z z H^m dz \,.$$

To get a quick estimate of the solution, we consider the first term of $H(z) = H(0) = D$ near the wall, then $\phi = z^2 D^m/2$ and we get

$$H = D + (1-D)\,\mathrm{erf}[z(D^m/2)^{\frac{1}{2}}] \,. \qquad \text{(A-2)}$$

The inner expansion of this outer solution (A-2) as $t \to 0$ is

$$\begin{aligned} H = D_0 &+ \sigma_\infty^{\frac{1}{2}} [D_1 + (2D_0^m/\pi)^{\frac{1}{2}} (D_0 - 1)(\beta_0 - \eta)] \\ &+ \sigma_\infty [D_2 + (2D_0^m/\pi)^{\frac{1}{2}} D_1 \{1 + m(D_0 - 1)/(2D_0)\} (\beta_0 - \eta) \\ &+ \beta_1 (2D_0^m/\pi)^{\frac{1}{2}} (D_0 - 1)] + O(\sigma^{\frac{3}{2}}) \,. \end{aligned} \qquad \text{(A-3)}$$

For an insulated wall matching (A-3) and (19), it is seen that the expressions for the recovery factor and skin friction are the same as (28) and (29) respectively. For the heat transfer case, matching (A-3) and (35) gives an expression for A which is the same as (42) while for α we obtain

$$\alpha = (m+1)\, T_w^{m/[2(1+m)]} [T_w^{m/(1+m)} - T_w] \sigma_\infty^{\frac{1}{2}} \,. \qquad \text{(A-4)}$$

The coefficient of skin friction can be obtained by substituting α given by (A-4) in (43).

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